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Mode conversion in the Gulf of Guinea

By A. N. KAUFMAN[1,2], J. J. MOREHEAD[1,2], A. J. BRIZARD[1] AND E. R. TRACY[3]

[1] Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA

[2] Physics Department, University of California, Berkeley, CA 94720-7300, USA

[3] Physics Department, College of William & Mary, Williamsburg, VA 23187-8795, USA

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Linear mode conversion is the partial transfer of wave energy from one wave type (a) to another (b) in a weakly non-uniform background state. For propagation in one dimension (x), the local wavenumber k_x^j of each wave ($j = a, b$) varies with x; if these are equal at some x_R, the waves are locally in phase, and resonant energy transfer can occur. We model wave propagation in the Gulf of Guinea, where wave a is an equatorially trapped Rossby–gravity (Yanai) wave, and wave b is a coastal Kelvin wave along the (zonal) north coast of the Gulf, both propagating in zonal coordinate x. The coupling of the waves is due to the overlap of their eigenfunctions (normal modes in y, the meridional coordinate). We derive coupled mode equations from a variational principle, and obtain an analytic expression for the wave-energy conversion coefficient, in terms of the wave frequency and the scale length of the thermocline depth.

1. Introduction

When linear waves of two different types propagate in a weakly non-uniform medium, there may be local regions where the waves are in phase, or linearly *resonant*. Then linear coupling between the waves produces a local energy transfer, whose magnitude depends also on the spatial scale of the non-uniformity.

Since this phenomenon is prevalent in plasma physics, a general formulation has been developed (Kaufman & Friedland 1987; Tracy & Kaufman 1993; Flynn & Littlejohn 1994), which we here apply to an oceanographic problem. Some years ago it was pointed out by Cane & Sarachik (1979) and by Philander (1977) (see also Moore 1968) that an interesting mode coupling appears in the Gulf of Guinea, whose northern coast is approximately zonal at 5° N. (See figure 1.) The Cane–Sarachik analysis utilizes a shallow-water model of the upper layer of the ocean, linearized about a reference state having a *uniform* thermocline depth H, and studies linear normal modes of the upper layer. The wave field (u, v, h) consists of the vertically uniform horizontal velocity $\boldsymbol{u} = u\,\hat{\boldsymbol{e}}_x + v\,\hat{\boldsymbol{e}}_y$ of the upper layer and the downward displacement h of the thermocline. The horizontal coordinates are $x =$ eastward and $y =$ northward from the Equator.

In the absence of the northern boundary at y_N, one finds (Cushman-Roisin 1994, Ch. 19; Moore & Philander 1977) an equatorially trapped mixed Rossby–gravity wave (or *Yanai* wave), with Gaussian dependence on y/R_e, where R_e is the equatorial Rossby radius $R_e = (c/\beta)^{1/2}$. (The equatorial β-plane model has Coriolis parameter $f(y) \equiv \beta\, y$, and characteristic ‘shallow-water-wave’ speed $c \equiv (g'H)^{1/2}$, in terms of

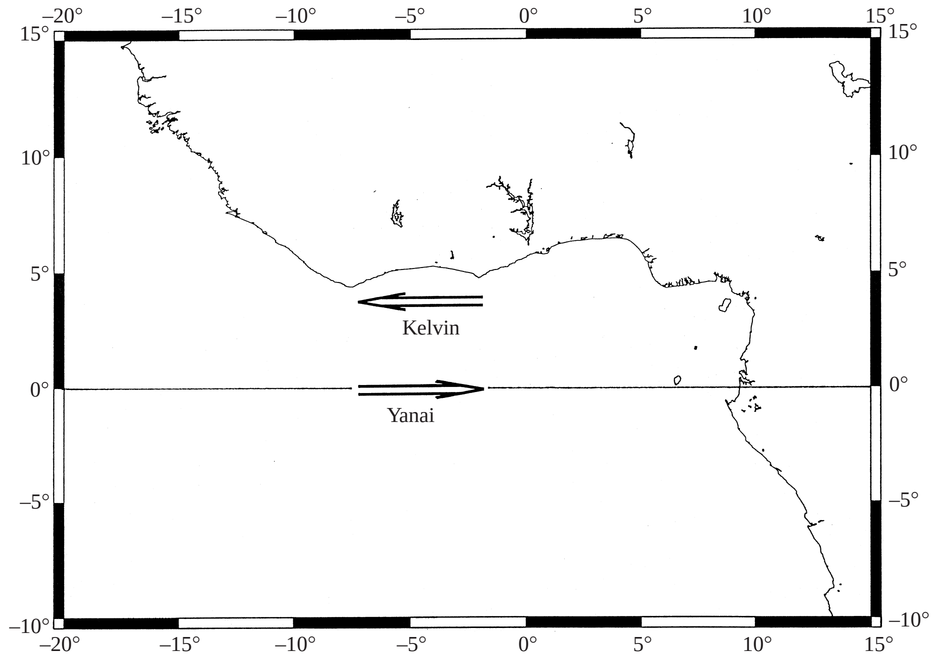

FIGURE 1. The Gulf of Guinea with a Yanai wave propagating eastwards along the equator and a Kelvin wave propagating westwards along the south coast of West Africa.

the reduced gravity $g' \equiv (\Delta\rho/\rho)g$, where $\Delta\rho$ is the density difference across the thermocline.) The Yanai dispersion relation $k_x^{\mathscr{Y}}(\omega)$ is

$$k_x^{\mathscr{Y}}(\omega) = \frac{\omega}{c} - \frac{\beta}{\omega}, \tag{1.1}$$

displayed non-dimensionally in figure 2 as

$$k_x^{\mathscr{Y}} R_e = \left(\frac{\omega}{(\beta c)^{1/2}}\right) - \left(\frac{\omega}{(\beta c)^{1/2}}\right)^{-1}. \tag{1.2}$$

(For nominal values, $H = 35\,\mathrm{m}$, $g' = 2\,\mathrm{cm\,s^{-2}}$, we have $c = 85\,\mathrm{cm\,s^{-1}}$. Then, with $\beta \approx 2.3 \times 10^{-13}\,\mathrm{cm^{-1}s^{-1}}$, we have $R_e \approx 190\,\mathrm{km}$, which is the characteristic reduced wavelength $\bar{\lambda} \equiv \lambda/2\pi \equiv k_x^{-1}$; and $(\beta c)^{1/2} \approx 4.4 \times 10^{-6}\,\mathrm{s^{-1}}$, giving a characteristic wave period $\tau = 2\pi/(\beta c)^{1/2} \approx 16.5$ days.)

Along the northern boundary at y_N, the f-plane model (f = constant $= f_N \equiv \beta\, y_N$) yields the coastally trapped Kelvin wave (Cushman-Roisin 1994, Sec. 6-2), a normal mode with exponential falloff in $(y_N - y)/R_N$, where R_N is the local Rossby radius at y_N, i.e. $R_N \equiv c/f_N$. Its dispersion relation is

$$k_x^{\mathscr{K}}(\omega) = -\omega/c \quad \text{or} \quad k_x^{\mathscr{K}} R_e = -\frac{\omega}{(\beta c)^{1/2}}, \tag{1.3}$$

also displayed in figure 2. (For $y_N = 5^\circ\,\mathrm{N} \approx 550\,\mathrm{km}$, we have $f_N = 12 \times 10^{-6}\,\mathrm{s^{-1}}$, or an inertial period $\tau_N = 2\pi/f_N \approx 6$ days. The Rossby radius is thus $R_N \approx 70\,\mathrm{km}$.)

In figure 3, we show the Yanai and Kelvin eigenfunctions, noting that they are normal modes for slightly different models. Since $R_e + R_N \approx 260\,\mathrm{km}$ is less than $y_N \approx 550\,\mathrm{km}$, their overlap is exponentially small, indicating that the physical coupling

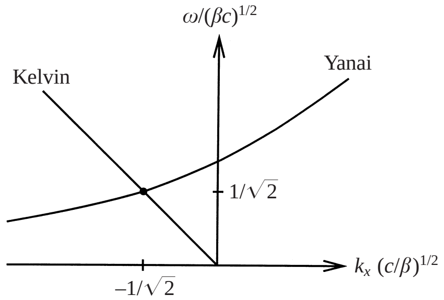

FIGURE 2. Dispersion curves $\omega(k_x)$ for Yanai and coastal Kelvin waves in dimensionless form. The curves cross at $\omega_R/(\beta c)^{1/2} = 1/\sqrt{2}$, $k_R(c/\beta)^{1/2} = -1/\sqrt{2}$.

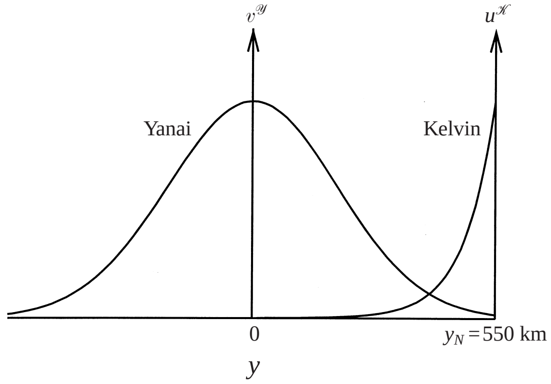

FIGURE 3. Normal-mode eigenfunctions for the Yanai (v vs. y) and Kelvin (u vs. y) waves, for $R_e/y_N \equiv \sqrt{2}\omega_R/f_N = 0.35$ and $R_N/y_N \equiv 2(\omega_R/f_N)^2 = 0.12$. Note the small overlap, where coupling occurs, resulting in conversion.

between the modes is weak. This weak coupling is effective, however, if the Yanai and Kelvin waves are *in phase*, i.e. if they have nearly the same frequency ω and wavenumber k_x.

The condition for linear resonance is obtained by equating the wavenumbers k_x^j of the two waves ($j = \mathscr{Y}, \mathscr{K}$), for equal ω and given c:

$$k_x^{\mathscr{Y}}(\omega) = k_x^{\mathscr{K}}(\omega). \tag{1.4}$$

From (1.1) and (1.3), we solve for ω to obtain the resonant frequency ω_R and resonant wavenumber k_x^R, in terms of c:

$$\frac{\omega_R}{(\beta c)^{1/2}} = \frac{1}{\sqrt{2}}, \quad k_x^R \left(\frac{c}{\beta}\right)^{1/2} = -\frac{1}{\sqrt{2}}. \tag{1.5}$$

(For $c = 85\,\mathrm{cm\,s^{-1}}$, the resonant period is $\tau_R \equiv 2\pi/\omega_R \approx 23$ days, and the resonant reduced wavelength is $\bar{\lambda}_R \equiv |k_x^R|^{-1} \approx 365\,\mathrm{km}$.) Thus, if c is uniform in x, a Yanai wave whose frequency is not near ω_R is not resonantly coupled to the Kelvin wave at that frequency.

However, for non-uniform $c(x)$, but not allowing y-dependence of c, a Yanai wave of frequency ω and x-dependent k_x may pass through resonance with the Kelvin wave at that frequency. The zonal non-uniformity of the thermocline depth $H(x)$, and thus of $c(x)$, is a characteristic feature of equatorial oceans. (For the Atlantic, see Philander 1990 (figure 2.16, from Merle 1980), for seasonal averages of this

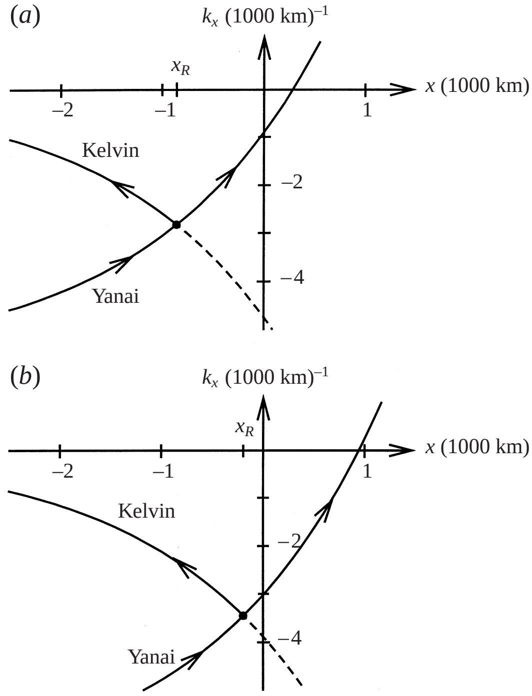

FIGURE 4. Dispersion curves (or rays) in (k_x, x) space for two representative wave periods, (*a*) 18 days and (*b*) 22 days. Conversion from Yanai to Kelvin occurs at the point x_R where the Yanai ray crosses the Kelvin dispersion curve.

eastward shallowing.) The local dispersion relations of the two waves now exhibit the x-dependence:

$$k_x^{\mathscr{Y}}(x;\omega) = \omega/c(x) - \beta/\omega \tag{1.6a}$$

$$k_x^{\mathscr{K}}(x;\omega) = -\omega/c(x). \tag{1.6b}$$

In figure 4, we display $k_x^{\mathscr{Y}}(x)$ and $k_x^{\mathscr{K}}(x)$ for a model thermocline, $c(x) = c_0 \exp(-x/L_c)$, where $c_0 = 85\,\mathrm{cm\,s^{-1}}$ and $L_c = 3\,y_N = 1650\,\mathrm{km}$, and for two representative frequencies. The arrows indicate the direction of the group velocity, i.e. of energy flux of each wave. Thus the dispersion curves can be considered as rays $[x(t), k_x(t)]$ in (x, k_x) phase space.

We see that the rays cross at the ω-dependent position $x_R(\omega)$, determined by again equating $k_x^{\mathscr{Y}} = k_x^{\mathscr{K}}$, but now solving for c in terms of fixed ω:

$$c(x_R) \equiv c_R(\omega) = 2\omega^2/\beta. \tag{1.7}$$

(This is of course equivalent to (1.5).) In the neighbourhood of the resonant crossing, then, an incident Yanai wave can convert a fraction C of its energy flux to the Kelvin wave, the remainder T being transmitted.

This paper is devoted to the analytic formulation of this process, resulting in an explicit expression for T and C, in terms of the wave frequency ω (relative to f_N) and the scale length L_c (relative to y_N). In § 2 we represent the set of linearized evolution equations for the wave field in non-uniform $H(x)$ as a Schrödinger-type equation (2.15) for the three-component field $\boldsymbol{\psi}$ (2.12). The evolution matrix $\boldsymbol{L}$ (2.16) is shown to be a Hermitian operator for a Hilbert space, whose metric (2.14) is obtained from the expression (2.7) for wave energy. This allows the evolution to be expressed as a variational principle, for time-dependent or fixed-frequency fields.

In § 3 we use this variational principle for a single wave (either Yanai or Kelvin), in the non-resonant case. By specifying the wave polarization $(u : v : h)$, we obtain the appropriate local dispersion relation, remarkably without explicitly choosing the normal-mode y-dependence. The energy conservation law yields the x-dependence of wave amplitude.

In § 4 we apply the variational principle to the two-wave problem, obtaining coupled equations for their amplitudes in standard form. For the case of uniform H, we recover the Cane–Sarachik result for the splitting of the frequency eigenvalues. In the non-uniform case, we obtain explicit expressions for the transmission and conversion coefficients. We find that the characteristic wave period for significant conversion (for $C \approx 0.5$) is about 3 weeks.

In § 5 we discuss possible extensions of the present work, and consider the observability of this process in the Gulf of Guinea.

We should point out that (in addition to the process studied here) conversion can occur as well at a discontinuity, such as the east coast of the Gulf of Guinea, where complete conversion of a Yanai wave to coastal Kelvin waves occurs (Moore 1968).

2. Variational principle

Linear mode conversion is the process whereby a wave a of one type transfers a fraction of its energy to a second wave b (of different type). The two waves have the same constant frequency ω (for a time-independent medium), while their weakly non-uniform wavenumbers $k_a(x), k_b(x)$ are locally equal (for a spatially non-uniform medium) at some location x_R. There the waves are locally in phase, resulting in linear resonance and energy transfer This process has been extensively studied for eikonal (quasi-plane) waves in the context of plasma physics. Here we extend the analysis to waves which are eikonal in x only, but are quasi-normal modes in y.

This process can occur in the Gulf of Guinea, whose north coast we model as zonal at 5° N. We employ the linearized $1\frac{1}{2}$-layer model (Cushman-Roisin 1994, p. 178) for the evolution of the vertically uniform horizontal velocity $[u(x, y, t), v(x, y, t)]$ in the upper mixed layer of depth $H(x)$. The x-dependence ($L_H \equiv |\mathrm{d} \ln H/\mathrm{d}x|^{-1} \approx 1000\,\mathrm{km}$) of the thermocline is essential for the mode-conversion process.

The nonlinear continuity equation for the upper layer of total depth $H(x)+h(x, y, t)$, is

$$\partial_t(H + h) = -\partial_x[(H + h)u] - \partial_y[(H + h)v]. \tag{2.1}$$

We linearize about a reference state with depth $H(x)$, non-uniform in x but independent of y and t, and with zero flow. Then the linearized continuity equation is

$$\partial_t h = -\partial_x (Hu) - \partial_y (Hv). \tag{2.2}$$

The linearized momentum equations are

$$\left.\begin{aligned} \partial_t u - fv &= -g'\partial_x h, \\ \partial_t v + fu &= -g'\partial_y h. \end{aligned}\right\} \tag{2.3}$$

The evolution equations (2.2), (2.3) satisfy a quadratic energy conservation law:

$$\frac{\partial \mathscr{E}(x, y; t)}{\partial t} = -\,\nabla \cdot \boldsymbol{S}(x, y; t), \tag{2.4}$$

where $\mathscr{E}$ is the wave energy per two-dimensional area:

$$\mathscr{E}(x, y; t) \equiv \tfrac{1}{2}\,\rho_0 H(x)\,(u^2 + v^2) + \tfrac{1}{2}\,\rho_0 g'\,h^2, \tag{2.5}$$

and $\boldsymbol{S}$ is the two-dimensional wave-energy flux density:

$$\boldsymbol{S}(x, y; t) \equiv \rho_0 c^2(x) h\,\boldsymbol{u}. \tag{2.6}$$

For fixed frequency ω, with $u(x, y; t) = u(x, y)\,\mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.}$, etc., the period-averaged energy density is, from (2.5),

$$\bar{\mathscr{E}}(x, y) \equiv \rho_0 H(x)\,(|u(x, y)|^2 + |v(x, y)|^2) + \rho_0 g'\,|h(x, y)|^2, \tag{2.7}$$

and the period-averaged energy flux is, from (2.6),

$$\bar{\boldsymbol{S}}(x, y) = \rho_0\,c^2(x)(h^* \boldsymbol{u} + h\boldsymbol{u}^*), \tag{2.8}$$

while the energy-conservation law (2.4) becomes

$$0 = \nabla \cdot \bar{\boldsymbol{S}}. \tag{2.9}$$

On integration over y, with the boundary condition $v(x, y_N) = 0$, (2.9) becomes the one-dimensional energy-conservation law:

$$0 = \frac{\mathrm{d}}{\mathrm{d}x} S(x), \tag{2.10}$$

where

$$S(x) \equiv \int_{-\infty}^{y_N} \mathrm{d}y\,\bar{\boldsymbol{S}}(x, y) \cdot \hat{\boldsymbol{x}} \tag{2.11}$$

is the total period-averaged energy flux in the x-direction.

To obtain a variational principle, we shall express the set of evolution equations as a Schrödinger-like equation, with a Hermitian evolution operator acting on a Hilbert space of states. First, expressing the complex-valued (x, y)-dependent state (u, v, h) as a column vector

$$\boldsymbol{\psi}(x, y) \equiv \begin{pmatrix} u \\ v \\ h \end{pmatrix} (x, y), \tag{2.12}$$

we write the energy density (2.7) as a quadratic form in the state $\boldsymbol{\psi}$:

$$\mathscr{E}(\boldsymbol{\psi}) \equiv \boldsymbol{\psi}^\dagger \cdot \boldsymbol{M} \cdot \boldsymbol{\psi}, \tag{2.13}$$

where

$$\boldsymbol{M} \equiv \begin{pmatrix} \rho_0 H(x) & 0 & 0 \\ 0 & \rho_0 H(x) & 0 \\ 0 & 0 & \rho_0\,g' \end{pmatrix} \tag{2.14}$$

is a positive-definite (x-dependent) metric matrix. (In (2.13), $\boldsymbol{\psi}^\dagger$ is the row vector (u^*, v^*, h^*).)

Our approach to linear conversion is based on a variational formulation of the linear evolution equations, (2.2) and (2.3), which we express as

$$\mathrm{i}\,\partial_t \boldsymbol{\psi} = \boldsymbol{L} \cdot \boldsymbol{\psi}\,, \tag{2.15}$$

where

$$\boldsymbol{L} \equiv \begin{pmatrix} 0 & \mathrm{i}\,\beta\,y & g'\hat{k}_x \\ -\mathrm{i}\,\beta\,y & 0 & g'\hat{k}_y \\ \hat{k}_x\,H(x) & \hat{k}_y\,H(x) & 0 \end{pmatrix}. \tag{2.16}$$

The evolution matrix $\boldsymbol{L}$ is an operator, with $\hat{k}_x \equiv -\mathrm{i}\,\partial_x$, $\hat{k}_y \equiv -\mathrm{i}\,\partial_y$. (Note that $\hat{k}_x H(x) \neq H(x)\hat{k}_x$.)

We wish to consider the set of three-component fields $\boldsymbol{\psi}(x, y)$ as elements of a Hilbert space. Accordingly, using the metric $\boldsymbol{M}$, we define an inner product:

$$\langle \boldsymbol{\psi}_1, \boldsymbol{\psi}_2 \rangle \equiv \int \mathrm{d}x \int_{-\infty}^{y_N} \mathrm{d}y\, \boldsymbol{\psi}_1^{\dagger} \cdot \boldsymbol{M} \cdot \boldsymbol{\psi}_2. \tag{2.17}$$

Thus the norm of a state $\boldsymbol{\psi}$ is its total wave energy, by (2.13):

$$\|\boldsymbol{\psi}\|^2 \equiv \langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle = \iint \mathrm{d}x\,\mathrm{d}y\,\bar{\mathscr{E}}(\boldsymbol{\psi}). \tag{2.18}$$

We can now verify that $\boldsymbol{L}$ is Hermitian with respect to the inner product (2.17), i.e.

$$\langle \boldsymbol{\psi}_1, \boldsymbol{L} \cdot \boldsymbol{\psi}_2 \rangle = \langle \boldsymbol{L} \cdot \boldsymbol{\psi}_1, \boldsymbol{\psi}_2 \rangle, \tag{2.19}$$

upon imposing the boundary condition $v(x, y_N, t) = 0$. We then form the variational functional

$$\mathscr{A}'(\boldsymbol{\psi}) \equiv \int \mathrm{d}t\, \langle \boldsymbol{\psi}, (\mathrm{i}\,\partial_t \boldsymbol{\psi} - \boldsymbol{L} \cdot \boldsymbol{\psi}) \rangle \tag{2.20}$$

on states $\boldsymbol{\psi}(t)$, and obtain (2.15) by demanding stationarity of $\mathscr{A}'$ with respect to arbitrary variation of $\boldsymbol{\psi}$. Alternatively, for time-dependence $\exp(-\mathrm{i}\,\omega\, t)$, we define

$$\begin{aligned} \mathscr{A}(\boldsymbol{\psi}) &\equiv \langle \boldsymbol{\psi}, (\omega \boldsymbol{I} - \boldsymbol{L}) \cdot \boldsymbol{\psi} \rangle \\ &\equiv \iint \mathrm{d}x\,\mathrm{d}y\, \boldsymbol{\psi}^{\dagger} \cdot \boldsymbol{D} \cdot \boldsymbol{\psi} \end{aligned} \tag{2.21}$$

where the 'dispersion' matrix operator $\boldsymbol{D}$ is defined as

$$\boldsymbol{D} \equiv \boldsymbol{M} \cdot (\omega \boldsymbol{I} - \boldsymbol{L}) = \rho_0 \begin{pmatrix} \omega H & -\mathrm{i}\beta\, y H & -\hat{k}_x c^2 \\ \mathrm{i}\beta\, y H & \omega H & -\hat{k}_y c^2 \\ -c^2 \hat{k}_x & -c^2 \hat{k}_y & \omega\, g' \end{pmatrix}, \tag{2.22}$$

which is manifestly Hermitian in the conventional sense. On varying $\mathscr{A}$, we obtain

$$\boldsymbol{D} \cdot \boldsymbol{\psi} = 0, \tag{2.23}$$

equivalent to (2.15) when ω is fixed.

Rather than solving (2.23), our approach is to use the variational principle (2.21), with $\boldsymbol{\psi}$ now constrained to a physically motivated subset of the full set of states. Thus, in § 3, we allow $\boldsymbol{\psi}$ to represent a single wave only, either a Yanai wave or a coastal Kelvin wave. Then, in § 4, the state $\boldsymbol{\psi}$ is a superposition of the two waves, and locally resonant coupling leads to linear conversion.

While our variational principle represents dynamics for the β-plane bounded at y_N, the Yanai and Kelvin waves are not normal modes of this system. Rather, the Yanai wave is a normal mode of the unbounded β-plane, and the Kelvin wave is a normal mode of the f-plane bounded at y_N. From their respective models, we obtain their polarizations in order to define these waves for the bounded β-plane.

3. Single wave

3.1. *Yanai wave*

For each single wave, we begin by considering the case of c uniform in x, so that the wave field $\boldsymbol{\psi}$ can have the x-dependence $\mathrm{e}^{\mathrm{i}kx}$. For the Yanai wave in the bounded

β-plane, we choose its polarization (the ratio $u : v : h$) to be that of an unbounded ($y_N \to \infty$) Yanai wave:

$$u : v : h = \mathrm{i}\,\omega y : c : \mathrm{i}\,\omega y c/g'. \tag{3.1}$$

To derive (3.1), we use the v-eigenfunction, $v \propto \exp[-\frac{1}{2}(y/R_e)^2]$, to replace $\partial_y v$ in (2.2) by $-(y/R_e^2)v$. Then, with $\partial_t = -\mathrm{i}\,\omega$ and $\partial_x = \mathrm{i}k_x$, we eliminate h from (2.2) and (2.3) and solve for u/v, obtaining

$$\frac{u}{v} = \frac{\mathrm{i}\beta y}{\omega - k_x c}. \tag{3.2}$$

The dispersion relation (1.1) implies that $\omega - k_x c = \beta c/\omega$; using this in (3.2) then yields $u/v = \mathrm{i}\,\omega y/c$. Then (2.3) with (1.1) yields $u/h = g'/c$.

Thus we take, as the Yanai trial vector,

$$\boldsymbol{\psi}^{\mathscr{Y}}(x, y) = \tilde{a}\,\mathrm{e}^{\mathrm{i}kx} \begin{pmatrix} \mathrm{i}\,\omega y \\ c \\ \mathrm{i}\,\omega y c/g' \end{pmatrix} F(y), \tag{3.3}$$

where $\tilde{a}$ is its (real, constant) dimensionless amplitude, $\tilde{a} \equiv |v(x, y = 0)|/c$, and $F(y)$ is at first an arbitrary dimensionless function, subject only to the boundary conditions $F(y_N) = 0 = F(-\infty)$ and the normalization $F(0) = 1$.

On substituting (3.3) into (2.21) it is straightforward to obtain

$$\mathscr{A}(\boldsymbol{\psi}^{\mathscr{Y}}) = \int \mathrm{d}x\, D_a(k, \omega)\, \tilde{a}^2, \tag{3.4}$$

with the Yanai dispersion function $D_a(k, \omega)$:

$$D_a(k, \omega) = \left(\frac{\omega}{c} - \frac{\beta}{\omega} - k\right) \bar{D}_a, \tag{3.5}$$

$$\bar{D}_a = (2\omega^2 \rho_0 c^3/g') \int_{-\infty}^{y_N} \mathrm{d}y\, y^2 |F(y)|^2. \tag{3.6}$$

On varying (3.4) with respect to the amplitude $\tilde{a}$, we obtain $D_a(k, \omega) = 0$, yielding the Yanai dispersion relation, (1.1). Thus (1.1) results here solely from the choice of polarization.

Next we allow for weak x-dependence of H (and thus c), and replace $\tilde{a}\,\mathrm{e}^{\mathrm{i}kx}$, in (3.3), with the eikonal (WKB) expression

$$\tilde{a}(x)\,\mathrm{e}^{\mathrm{i}\theta(x)} \tag{3.7}$$

with the local wavenumber $k(x) \equiv \mathrm{d}\theta/\mathrm{d}x$. We assume formally that $kL \gg 1$, where L represents the scale length of the spatial variation of H, and thus also of c, $\tilde{a}$, and k. Therefore the operator $\hat{k}_x$ in (2.22) acts, to lowest approximation, only on $\mathrm{e}^{\mathrm{i}\theta(x)}$, and can thus be replaced by $k(x)$. In (3.3) we retain the same polarization locally replacing c by $c(x)$, while the y-dependence of F generalizes to include slow x-dependence of $F(x, y)$. Thus (3.3) is replaced by

$$\boldsymbol{\psi}^{\mathscr{Y}}(x, y) = \tilde{a}(x)\,\mathrm{e}^{\mathrm{i}\theta(x)} \begin{pmatrix} \mathrm{i}\,\omega y \\ c(x) \\ \mathrm{i}\,\omega y c(x)/g' \end{pmatrix} F(x, y). \tag{3.8}$$

Substituting (3.8) into (2.21), we obtain the generalization of (3.4):

$$\mathscr{A}(\psi^{\mathscr{Y}}) = \int \mathrm{d}x\, D_a(x,k)\, \tilde{a}^2(x), \tag{3.9}$$

with

$$D_a(x,k) = \left(\frac{\omega}{c(x)} - \frac{\beta}{\omega} - k(x)\right) \bar{D}_a(x), \tag{3.10}$$

$$\bar{D}_a(x) = (2\omega^2 \rho_0 c^3(x)/g') \int \mathrm{d}y\, y^2\, |F(x,y)|^2. \tag{3.11}$$

On varying (3.9) with respect to the x-dependent amplitude $\tilde{a}(x)$, we now obtain $D_a(x,k(x)) = 0$, and the local Yanai dispersion relation (1.6a). Varying (3.9) with respect to the phase $\theta(x)$, which appears only in $k \equiv \mathrm{d}\theta/\mathrm{d}x$, we have

$$\delta\mathscr{A} = \int \mathrm{d}x\, \tilde{a}^2(x) \left(\partial D_a/\partial k\right)\, \delta k(x), \tag{3.12}$$

with $\delta k(x) \equiv \mathrm{d}\,\delta\theta(x)/\mathrm{d}x$. On integrating by parts, we obtain

$$\delta\mathscr{A} = \int \mathrm{d}x\, \delta\theta(x) \frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{\partial D_a}{\partial k}(x,k)\, \tilde{a}^2(x)\right]. \tag{3.13}$$

With $\delta\mathscr{A} = 0$ for all $\delta\theta(x)$, we obtain the conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{\partial D_a}{\partial k}(x)\, \tilde{a}^2(x)\right] = 0. \tag{3.14}$$

But by (3.10), $-\partial D_a/\partial k = \bar{D}_a(x)$, so (3.14) yields

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\bar{D}_a(x)\, \tilde{a}^2(x)\right] = 0, \tag{3.15}$$

which determines the x-variation of $\tilde{a}(x)$.

The physical interpretation of (3.15) is that it represents energy conservation. To see this, we return to the general expression (2.21), replace $\hat{k}_x$ in (2.22) by $k(x)$ and, on varying (2.21) with respect to $\theta(x)$, obtain

$$0 = \delta\mathscr{A} = \int \mathrm{d}x\, \delta\theta(x) \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{-\infty}^{y_N} \mathrm{d}y\, \rho_0\, c^2(x)(h^* u + h u^*)(x,y)\right]. \tag{3.16}$$

We recognize the bracketed expression in (3.16) as $S(x)$, from (2.11) and (2.8). Thus (3.16) is

$$\delta\mathscr{A} = \int \mathrm{d}x\, \delta\theta(x)\, \mathrm{d}S(x)/\mathrm{d}x. \tag{3.17}$$

Comparing this to (3.13), we identify (3.15) as the energy-conservation law (2.10), with the Yanai energy flux:

$$S^{\mathscr{Y}} = \bar{D}_a(x)\, \tilde{a}^2(x). \tag{3.18}$$

To evaluate $\bar{D}_a(x)$, we first choose for the function $F(y)$ in (3.3), that appropriate for $y_N \to \infty$, but truncated at y_N:

$$F(y) = \begin{cases} \exp\left[-\frac{1}{2}\left(y/R_e\right)^2\right], & y < y_N \\ 0, & y \geqslant y_N. \end{cases} \tag{3.19}$$

Then for non-uniform $c(x)$ we use the local equatorial Rossby radius:

$$R_e(x) = (c(x)/\beta)^{1/2}, \tag{3.20}$$

in (3.19), and thus take

$$F(x,y) = \exp\left[-\frac{1}{2}\left(\frac{y}{R_e(x)}\right)^2\right] \quad \text{for } y < y_N. \tag{3.21}$$

Substituting (3.21) into (3.11), we obtain

$$\bar{D}_a(x) = (\pi^{1/2}\,\omega^2\,\rho_0/g'\beta^{3/2})\,c^{9/2}(x), \tag{3.22}$$

neglecting the exponentially small correction of order $\exp[-(y_N/R_e)^2]$. (Recall that, for our nominal parameters, we have $y_N \approx 2R_e$, so the correction is of order $e^{-4} \approx 2\%$.) Inserting (3.22) into (3.15), we obtain the Yanai amplitude dependence on $c(x)$:

$$\tilde{a}(x) \propto c^{-9/4}(x). \tag{3.23}$$

Thus, as the Yanai wave travels eastward, its dimensionless amplitude increases as the layer depth decreases: $\tilde{a}(x) \propto H^{-9/8}(x)$. Since the thermocline displacement h is proportional to $c\tilde{a}$ (see (3.3)), it increases as $h(x) \propto H^{-5/8}(x)$.

3.2. *Coastal Kelvin wave*

We next perform the analogous calculation for a coastal Kelvin wave. For the case of uniform c, we take

$$\boldsymbol{\psi}^{\mathscr{K}}(x,y) = \tilde{b}\,e^{ikx}\begin{pmatrix} c \\ 0 \\ -H \end{pmatrix} G(y), \tag{3.24}$$

where the polarization

$$u : v : h = c : 0 : -H \tag{3.25}$$

is the standard result for the f-plane model. At first, we do not specify the dimensionless $G(y)$, except for its normalization $G(y_N) = 1$. Substituting (3.24) into (2.21), we obtain

$$\mathscr{A}(\boldsymbol{\psi}^{\mathscr{K}}) = \int dx\, D_b(k,\omega)\,\tilde{b}^2, \tag{3.26}$$

with

$$D_b(k,\omega) = \left(\frac{\omega}{c} + k\right)\bar{D}_b, \tag{3.27}$$

$$\bar{D}_b = (2\rho_0 c^5/g')\int_{-\infty}^{y_N} dy\,|G(y)|^2. \tag{3.28}$$

Varying (3.26) with respect to $\tilde{b}$, we obtain $D_b(k,\omega) = 0$, leading to the standard Kelvin dispersion relation (1.3).

Then we allow for weak x-dependence, and replace $\tilde{b}\,e^{ikx}$ by $\tilde{b}\,e^{i\theta(x)}$:

$$\boldsymbol{\psi}^{\mathscr{K}}(x,y) = \tilde{b}(x)\,e^{i\theta(x)} \begin{pmatrix} c(x) \\ 0 \\ -H(x) \end{pmatrix} G(x,y), \tag{3.29}$$

where the local polarization

$$u : v : h = c(x) : 0 : -H(x) \tag{3.30}$$

is that of the uniform case, and $G(x, y = y_N) = 1$. Substituting into (2.21), we obtain

$$\mathscr{A}(\boldsymbol{\psi}^{\mathscr{K}}) = \int \mathrm{d}x\, D_b(x,k)\,\tilde{b}^2(x), \tag{3.31}$$

with

$$D_b(x,k) = \left(\frac{\omega}{c(x)} + k(x)\right) \bar{D}_b(x), \tag{3.32}$$

$$\bar{D}_b(x) = \left(2\,\rho_0\,c^5(x)/g'\right) \int \mathrm{d}y\,|G(x,y)|^2. \tag{3.33}$$

Varying with respect to $\tilde{b}(x)$, we obtain the local Kelvin dispersion relation (1.6*b*), again without specifying $G(x,y)$. Varying with respect to $\theta(x)$, we obtain the energy conservation law:

$$\frac{\mathrm{d}S^{\mathscr{K}}}{\mathrm{d}x} = 0, \tag{3.34}$$

with the (constant) Kelvin energy flux:

$$S^{\mathscr{K}} = -\frac{\partial D_b}{\partial k}\,\tilde{b}^2 \tag{3.35}$$

$$= -\bar{D}_b(x)\,\tilde{b}^2(x). \tag{3.36}$$

The Kelvin energy flux is negative, and thus westward along the coast. To evaluate $\bar{D}_b(x)$, we take

$$G(x,y) = \exp\left[-\left(\frac{y_N - y}{R_N(x)}\right)\right], \tag{3.37}$$

appropriate to the f-plane model, where

$$R_N(x) = \frac{c(x)}{f_N} \tag{3.38}$$

is the local Rossby radius at y_N. Evaluating $\bar{D}_b$, we find

$$\bar{D}_b(x) = \left(\rho_0/g'\,f_N\right)\;c^6(x); \tag{3.39}$$

thus, from (3.34), (3.36), the Kelvin amplitude varies as

$$\tilde{b}(x) \propto c^{-3}(x). \tag{3.40}$$

As the Kelvin wave propagates westward, $c(x)$ increases, so the dimensionless amplitude $\tilde{b}(x)$ decreases, and the thermocline displacement decreases as $h(x) \propto c^{-1}$.

4. Linearly coupled Yanai and Kelvin waves

4.1. *Uniform H case*

We now consider a two-mode model consisting of the Yanai wave and the Kelvin wave, first for the case of uniform H, so that the x-dependence may be e^{ikx}:

$$\psi(x,y) = \psi^{\mathscr{Y}}(x,y) + \psi^{\mathscr{K}}(x,y). \tag{4.1}$$

Here $\psi^{\mathscr{Y}}$ is given by (3.3) (but with $\tilde{a}$ now allowed to be complex), and $\psi^{\mathscr{K}}$ by (3.24).

On substituting (4.1) into (2.21), we now obtain additional cross-terms, representing coupling:

$$\mathscr{A} = \int \mathrm{d}x\,[D_a|\tilde{a}|^2 + D_b|\tilde{b}|^2 + \eta\,\tilde{a}^*\tilde{b} + \eta^*\,\tilde{a}\tilde{b}^*], \tag{4.2}$$

with

$$\eta = \mathrm{i}\rho_0(c^4/g')\int_{-\infty}^{y_N} \mathrm{d}y\,F^*(y)\,(\beta\,y - c\,\partial_y)G(y). \tag{4.3}$$

On using (3.19) for $F(y)$ and (3.37) for $G(y)$, we obtain

$$\eta = -\mathrm{i}\,(\rho_0\,c^5/g')\exp\,[-\tfrac{1}{2}\,(y_N/R_e)^2], \tag{4.4}$$

with D_a given by (3.5) and (3.22), and D_b given by (3.32) and (3.39). Varying (4.2) with respect to $\tilde{a}$ and $\tilde{b}$, we obtain the coupled algebraic equations:

$$\left.\begin{aligned} D_a\,\tilde{a} + \eta\,\tilde{b} &= 0,\\ \eta^*\,\tilde{a} + D_b\,\tilde{b} &= 0. \end{aligned}\right\} \tag{4.5}$$

More explicitly,

$$\left.\begin{aligned} [k^{\mathscr{Y}}(\omega) - k]\bar{D}_a\,\tilde{a} + \eta\,\tilde{b} &= 0,\\ \eta^*\,\tilde{a} + [k - k^{\mathscr{K}}(\omega)]\,\bar{D}_b\,\tilde{b} &= 0. \end{aligned}\right\} \tag{4.6}$$

The condition for a solution, that the determinant vanishes:

$$[k^{\mathscr{Y}}(\omega) - k][k - k^{\mathscr{K}}(\omega)] = |\eta|^2/\bar{D}_a\,\bar{D}_b, \tag{4.7}$$

yields the dispersion relation $\omega(k)$ in the presense of coupling η. We limit ourselves to evaluating the splitting at the crossing: $k_R = -(\beta/2c)^{1/2}$, $\omega_R = (\beta c/2)^{1/2}$. (See (1.5) and figure 5*a*.) We expand $k^j(\omega) - k_R$ to first order in $(\omega - \omega_R)$:

$$k^j(\omega) - k_R = (\omega - \omega_R)(\mathrm{d}k^j/\mathrm{d}\omega), \tag{4.8}$$

and substitute this into (4.7). The result is that

$$\omega(k_R) = \omega_R \pm \Delta\omega, \tag{4.9}$$

with

$$\Delta\omega = |\eta|\left[\frac{c_g^a\,|c_g^b|}{\bar{D}_a\,\bar{D}_b}\right]^{1/2} = |\eta|\left[\frac{\partial D_a}{\partial\omega}\frac{\partial D_b}{\partial\omega}\right]^{-1/2}, \tag{4.10}$$

where $c_g^a \equiv (\mathrm{d}k_a/\mathrm{d}\omega)^{-1} = c/3$, $c_g^b \equiv (\mathrm{d}k_b/\mathrm{d}\omega)^{-1} = -c$. Evaluating (4.10), we obtain

$$\frac{\Delta\omega}{\omega_R} = \frac{2}{3^{1/2}\pi^{1/4}}\left(\frac{y_N}{R_e}\right)^{1/2}\exp[-\tfrac{1}{2}(y_N/R_e)^2], \tag{4.11}$$

in complete agreement with the results of Cane & Sarachik (1979), obtained by solving (2.23) exactly for the uniform case, and then studying the asymptotic limit

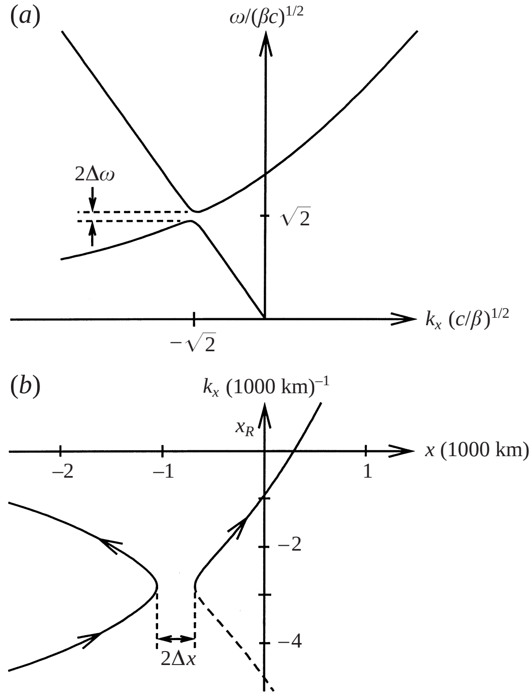

FIGURE 5. Dispersion curves for coupled Yanai and Kelvin waves: (*a*) ω vs. k_x for $c = 85\,\mathrm{cm\ s^{-1}}$; (*b*) k_x vs. x for wave period 18 days and thermocline scale length $L_c = 3y_N = 1650\,\mathrm{km}$ (as in figure 4*a*). In the gap k_x is complex, representing the tunnelling of the Yanai mode across the conversion region.

for $y_N \gg R_e$. This agreement serves as a validation for our approach based on the two-wave model (4.1).

4.2. *Non-uniform H case*

We now proceed to study the linear interaction of the two waves in the x-dependent thermocline, i.e. the conversion process. Here the eikonal assumption ($\mathrm{d}\ln\tilde{a}/\mathrm{d}x \ll k$) is dropped to allow for strong x-dependence of the amplitudes in this process, and $\tilde{a}(x)\,\mathrm{e}^{\mathrm{i}\theta(x)}$ of (3.8) is replaced by complex $a(x)$. Then $\hat{k}_x \equiv -\mathrm{i}\,\partial_x$ in (2.22) acts on the amplitude $a(x)$, but still not on $c(x)$. Then (3.9) becomes

$$\mathscr{A}(\boldsymbol{\psi}^{\mathscr{Y}}) = \int \mathrm{d}x\; a^*(x)\,\hat{D}_a(x,\hat{k}_x)\,a(x), \tag{4.12}$$

with (3.10) replaced by

$$\hat{D}_a = \left(\frac{\omega}{c(x)} - \frac{\beta}{\omega} - \hat{k}_x\right)\bar{D}_a. \tag{4.13}$$

Similarly (3.31) becomes

$$\mathscr{A}(\boldsymbol{\psi}^{\mathscr{K}}) = \int \mathrm{d}x\; b^*(x)\,\hat{D}_b(x,\hat{k}_x)\,b(x), \tag{4.14}$$

with

$$\hat{D}_b = \left(\frac{\omega}{c(x)} + \hat{k}_x\right)\bar{D}_b. \tag{4.15}$$

For the sum of the two waves, (4.2) becomes

$$\mathscr{A}(\boldsymbol{\psi}^{\mathscr{Y}} + \boldsymbol{\psi}^{\mathscr{K}}) = \int \mathrm{d}x\; [a^* \hat{D}_a a + b^* \hat{D}_b b + a^* \eta b + b^* \eta^* a]. \tag{4.16}$$

Varying with respect to a and b yields the coupled differential equations for $a(x)$ and $b(x)$:

$$\left.\begin{aligned} \hat{D}_a\, a(x) + \eta\, b(x) = 0, \\ \eta^*\, a(x) + \hat{D}_b\, b(x) = 0, \end{aligned}\right\} \tag{4.17}$$

similar in form to the algebraic equations (4.5).

To solve (4.17) in the neighbourhood of the conversion location x_R, we replace the function $\eta(x)$, given by (4.4), by its value at x_R: $\eta \equiv \eta(x_R)$; likewise we evaluate the factors $\bar{D}_a$, $\bar{D}_b$ of (3.22), (3.39) at x_R. Further, we linearize the x-dependent term $\omega/c(x)$ of (4.13), (4.15) with respect to x:

$$\frac{\omega}{c(x)} = \frac{\omega}{c_R}\left(1 + \frac{x - x_R}{L_c}\right), \tag{4.18}$$

where $L_c \equiv -(\mathrm{d}\ln c/\mathrm{d}x)^{-1}$ at x_R.

Now that (4.17) has been placed in the standard form (Kaufman & Friedland 1987) for linear mode conversion, where $\hat{D}_a$, $\hat{D}_b$ are linear in x, $\hat{k}_x$, while η is constant; we can solve (4.17) exactly, and quote the result for the transmission coefficient T, the fraction of Yanai energy flux not converted to the Kelvin wave:

$$T = \exp[-2\pi|\eta|^2/\mathscr{B}], \tag{4.19}$$

where

$$\begin{aligned} \mathscr{B} &\equiv |\{D_a, D_b\}| \\ &= |(\partial D_a/\partial x)(\partial D_b/\partial k_x) - (\partial D_a/\partial k_x)(\partial D_b/\partial x)| \end{aligned} \tag{4.20}$$

is the absolute value of the Poisson Bracket of the two dispersion functions, evaluated at x_R. From (4.20) and (4.4), we evaluate T, and obtain

$$T(\omega, L_c) = \exp\left[-(2\pi)^{1/2}\left(\frac{L_c}{y_N}\right)\left(\frac{f_N}{\omega}\right)^2 \mathrm{e}^{-(f_N/\omega)^2/2}\right], \tag{4.21}$$

valid when $\omega < f_N$ (so that the coupling is weak; when $\omega > f_N$, $T \approx 0$.) The reason that T depends on ω is via (1.7): a decrease of ω implies a decrease of c_R and thus of R_e and R_N, thereby decreasing $|\eta|$.

The energy conservation law (2.10) for the linearized version of (4.17) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[S^{\mathscr{Y}}(x) + S^{\mathscr{K}}(x)\right] = 0\,; \tag{4.22}$$

the coupling η, being independent of $\hat{k}_x$, does not contribute to the energy flux. Away from the conversion region $x \approx x_R$, (4.22) states that the energy flux west of x_R equals that east of x_R:

$$S^{\mathscr{Y}}(x \ll x_R) + S^{\mathscr{K}}(x \ll x_R) = S^{\mathscr{Y}}(x \gg x_R), \tag{4.23}$$

since $S^{\mathscr{K}}(x \gg x_R) = 0$ for no incoming Kelvin wave. (See figure 4.) Thus flux conservation can be expressed as

$$1 + (-C) = T, \tag{4.24}$$

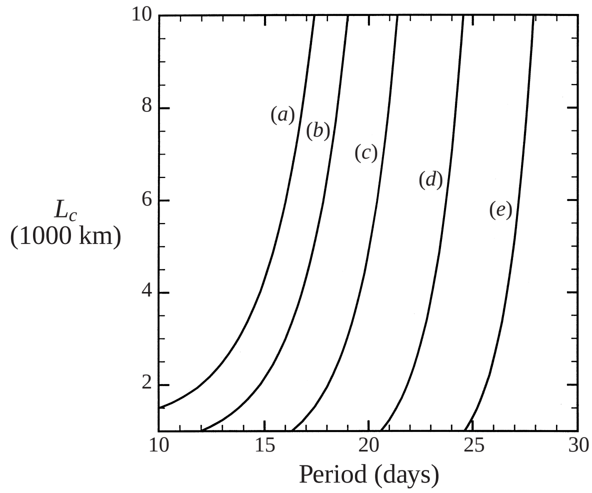

FIGURE 6. Contours of trnsmission coefficient as a function of wave period and thermocline scale length, for (*a*) $T = 0.01, C = 0.99$; (*b*) $T = 0.1, C = 0.9$; (*c*) $T = 0.5, C = 0.5$; (*d*) $T = 0.9, C = 0.1$; (*e*) $T = 0.99, C = 0.01$.

where the conversion coefficient C is the ratio of the magnitude of the converted Kelvin flux $|S^{\mathscr{K}}|$ to the incident Yanai flux. Hence energy conservation states that

$$C = 1 - T, \tag{4.25}$$

with T given by (4.19) or (4.21).

We display the level sets of C and T in figure 6, as functions of L_c and ω. We note the strong dependence on ω, and the weak dependence on L_c. Roughly, we may say that conversion is negligible for wave period greater than 3 weeks (curves (*d*) and (*e*) of figure 6), while for wave period less than 3 weeks conversion is substantial (curves (*a*), (*b*), and (*c*) of figure 6).

To estimate the width Δx of the conversion region (see figure 5*b*), we use the eikonal approximation $\hat{k}_x \to k(\omega)$ in (4.17), to obtain (4.7), but now with k replaced by $k(x)$, and $k^j(\omega)$ replaced by $k^j(x;\omega)$ (see (1.6)):

$$[k^{\mathscr{Y}}(x;\omega) - k(x)][k(x) - k^{\mathscr{K}}(x;\omega)] = |\eta|^2/\bar{D}_a\bar{D}_b\,. \tag{4.26}$$

Expanding about x_R, k_R, we find that $k(\omega)$ has caustics ($\mathrm{d}k/\mathrm{d}x = \infty$) at $x = x^R \pm \Delta x$, where

$$(\Delta x)^2 = \left|\frac{\mathrm{d}k^{\mathscr{Y}}}{\mathrm{d}x}\frac{\mathrm{d}k^{\mathscr{K}}}{\mathrm{d}x}\right|^{-1} |\eta|^2/\bar{D}_a\bar{D}_b. \tag{4.27}$$

We express $|\eta|^2$ in terms of T by (4.19):

$$|\eta|^2 = (\mathscr{B}/2\pi)\ln T^{-1}, \tag{4.28}$$

and so obtain, with (4.20),

$$(\Delta x)^2 = \left[\left|\frac{\mathrm{d}k^{\mathscr{Y}}}{\mathrm{d}x}\right|^{-1} + \left|\frac{\mathrm{d}k^{\mathscr{K}}}{\mathrm{d}x}\right|^{-1}\right](2\pi)^{-1}\ln T^{-1}\,. \tag{4.29}$$

Since $|\mathrm{d}k^j/\mathrm{d}x| = |k|/L = 2\pi/\lambda L$, we then obtain, for T neither very small nor very near 1,

$$\Delta x \approx (\lambda L)^{1/2}/2\pi. \tag{4.30}$$

Thus our local approximation, which assumes that $\Delta x \ll L$, is valid if $\lambda/2\pi \ll L$, as required by the eikonal theory.

Incidentally, the transmission exponent $|\eta|^2/\mathscr{B}$ can be expressed in terms of the frequency splitting $\Delta\omega$, by using the identity

$$\{D_a, D_b\} = \{\omega_a, \omega_b\} \frac{\partial D_a}{\partial \omega} \frac{\partial D_b}{\partial \omega}, \tag{4.31}$$

where $\omega_j(k_x, x)$ is the root of $D_j(k_x, x; \omega) = 0$. We find that

$$\frac{|\eta|^2}{\mathscr{B}} = \frac{(\Delta\omega)^2}{|\{\omega_a, \omega_b\}|}. \tag{4.32}$$

Thus information on the splitting of frequency eigenvalues allows direct determination of conversion.

5. Conclusions, observability, and extensions

We have shown that a Yanai wave propagating eastward (in terms of group velocity) on a zonally dependent thermocline, converts a fraction C of its energy flux to a westward-propagating coastal Kelvin wave, at a frequency-dependent location $x_R(\omega)$ where the respective wavenumbers match ($k^{\mathscr{Y}}(x_R) = k^{\mathscr{K}}(x_R)$). This conversion fraction is appreciable for wave periods of order 3 weeks or less.

Narrow-band Yanai waves have been observed in the Gulf of Guinea (Qiao & Weisberg 1995), and so have coastal Kelvin waves (Clarke & Battisti 1983; Picaut 1983). The latter may be produced by the conversion process studied here, but also by boundary conversion (Clarke 1983) at the eastern coast of the Gulf of Guinea. To distinguish between these two sources of the Kelvin waves, one could examine the time lag in the correlation between the Kelvin and Yanai surface displacements. Satellite altimeter data may provide such evidence for the conversion.

A number of extensions of the present study suggest themselves:

(1) A gradual conversion can occur as a result of the coast having a zonal dependence of its latitude. This situation is found at the western entrance to the Gulf of Guinea (figure 1), and also for the northern coast of New Guinea (south of the Equator). Here one could model the thermocline depth as uniform, and take into account the x-dependence of the coupling strength, varying with the overlap (figure 3). In contrast to the x-dependent-thermocline case studied here, conversion would arise only in the narrow frequency band at the eigenvalue splitting (figure 5*a*).

(2) One can analyse the conversion of a coastal Kelvin wave approaching $x_R(\omega)$ from the east into a Yanai wave. This process would have the same T and C as for the Yanai-to-Kelvin conversion. But if both incoming channels are occupied, interference occurs, and the full S-matrix for this process is needed (Tracy & Kaufman 1993, 1990).

(3) If the thermocline depth is non-monotonic (Philander 1990; Merle 1980), then succesive conversions can occur, with interesting effects of interference (Brizard *et al.* 1998; Liang *et al.* 1994).

(4) Slow seasonal time-dependence of the thermocline depth $H(x, t)$ can be treated

by using the Poisson bracket $\mathscr{B}$ (4.20) extended to four-dimensional phase space $(x, k_x; t, -\omega)$.

(5) Continuous stratification and vertically propagating modes (Philander 1977; Moore, Kloosterziel & Kessler 1998) can be treated by using the four-dimensional phase space $(x, k_x; z, k_z)$.

(6) Weak nonlinearity implies amplitude-dependence in the dispersion relations. This may lead to enhanced conversion due to phase-locking, i.e. the autoresonant phenomenon studied by Friedland (1995).

(7) If the reference state has a y-dependent flow $u^0(y)$, the Hermitian property of the evolution operator $\boldsymbol{L}$ (2.16) is lost. However, it may be possible to find a pseudo-Hermitian formulation, in terms of a pseudo-Hilbert space with indefinite metric (Brizard 1992; Brizard, Cook & Kaufman 1993).

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